

# PARABOLIC GEOMETRIES FOR PEOPLE THAT LIKE PICTURES

## LECTURE 5: INTERPRETING THE KILLING FORM (PART 1: AN ALGEBRAIC HEURISTIC)

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The Killing form is an *exceptionally* powerful idea in Lie theory. It is the key to understanding much of the structure theory of Lie algebras, and as such, it will be a vital part of our exploration into parabolic geometries.

Unfortunately, the Killing form is notoriously tricky to understand intuitively; many mathematicians even assume that it cannot be. This lecture and its sequel are the result of nearly a decade of trying to understand the intuition behind the Killing form; I am still not entirely satisfied—perhaps after another eight years I’ll have even better answers—but I hope that, by sharing this, I can help you avoid struggling with it as much as I did.

The lecture should proceed as follows:

- Review the definition of the Killing form
- Rediscover a convincing reason for the conic section terminology in the classification of elements of  $\mathfrak{sl}_2\mathbb{R}$
- Compare the Killing form on  $\mathfrak{sl}_2\mathbb{R}$  to the notion of eccentricity for conic sections
- Learn how to interpret the Killing form for general Lie algebras

As we said above, a fundamental understanding of the Killing form will be crucial for the lectures to come. In the next lecture, we will present the Killing form in a more geometric context, after which we will finally be ready to define parabolic subgroups.

### 1. INTRODUCTION

To start, let’s define the Killing form.

**Definition 1.1.** The *Killing form*  $\mathfrak{h}$  on a Lie algebra  $\mathfrak{g}$  is the symmetric bilinear form given by  $\mathfrak{h}(X, Y) := \text{tr}(\text{ad}_X \circ \text{ad}_Y)$ .

As an example, let us look at  $\mathfrak{sl}_2\mathbb{R}$ . For  $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \begin{bmatrix} z & y \\ x & -z \end{bmatrix} \in \mathfrak{sl}_2\mathbb{R}$ ,

$$\begin{aligned} \left[ \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \left[ \begin{bmatrix} z & y \\ x & -z \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right] \right] &= \left[ \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \begin{bmatrix} y & 0 \\ -2z & -y \end{bmatrix} \right] \\ &= \begin{bmatrix} -2bz & -2by \\ 2cy + 4az & 2bz \end{bmatrix}, \end{aligned}$$

Of course, given a pair of elements in a Lie algebra, we can compute the Killing form applied to that pair, but that ultimately just gives us a number. Visually, what does that number tell us?

$$\begin{aligned} \left[ \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \left[ \begin{bmatrix} z & y \\ x & -z \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right] \right] &= \left[ \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \begin{bmatrix} 0 & -2y \\ 2x & 0 \end{bmatrix} \right] \\ &= \begin{bmatrix} 2(bx + cy) & -4ay \\ -4ax & -2(bx + cy) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} \left[ \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \left[ \begin{bmatrix} z & y \\ x & -z \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] \right] &= \left[ \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \begin{bmatrix} -x & 2z \\ 0 & x \end{bmatrix} \right] \\ &= \begin{bmatrix} -2cz & 4az + 2bx \\ -2cx & 2cz \end{bmatrix}, \end{aligned}$$

so

$$\begin{aligned} \mathfrak{b}(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \begin{bmatrix} z & y \\ x & -z \end{bmatrix}) &= \text{tr} \left( \text{ad}_{\begin{bmatrix} a & b \\ c & -a \end{bmatrix}} \circ \text{ad}_{\begin{bmatrix} z & y \\ x & -z \end{bmatrix}} \right) \\ &= 2cy + 4ax + 2(bx + cy) + 4az + 2bx \\ &= 8az + 4(bx + cy). \end{aligned}$$

In particular, note that the elements  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , and  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  are orthogonal with respect to  $\mathfrak{b}$ , with

$$\mathfrak{b}(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = \mathfrak{b}(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = 8$$

and

$$\mathfrak{b}(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}) = -8,$$

so that  $\mathfrak{b}$  is nondegenerate on  $\mathfrak{sl}_2\mathbb{R}$  with signature  $(2, 1)$ .

Arguably one of the main reasons that the Killing form is so remarkably useful is that it is intrinsic to the Lie algebra itself, so that it does not depend on any particular description of its elements. Formally, this just means that the Killing form is automorphism-invariant.

**Proposition 1.2.** *The Killing form is invariant under automorphisms. In other words, if  $\phi$  is an automorphism of  $\mathfrak{g}$  and  $X, Y \in \mathfrak{g}$ , then  $\mathfrak{b}(\phi(X), \phi(Y)) = \mathfrak{b}(X, Y)$ .*

*Proof.* Because  $\phi$  is an automorphism,  $[\phi(X), Y] = \phi([X, \phi^{-1}(Y)])$ , hence  $\text{ad}_{\phi(X)} = \phi \circ \text{ad}_X \circ \phi^{-1}$ . Thus,

$$\begin{aligned} \mathfrak{b}(\phi(X), \phi(Y)) &= \text{tr}(\text{ad}_{\phi(X)} \circ \text{ad}_{\phi(Y)}) \\ &= \text{tr}((\phi \circ \text{ad}_X \circ \phi^{-1}) \circ (\phi \circ \text{ad}_Y \circ \phi^{-1})) \\ &= \text{tr}(\phi \circ (\text{ad}_X \circ \text{ad}_Y) \circ \phi^{-1}) \\ &= \text{tr}(\text{ad}_X \circ \text{ad}_Y) = \mathfrak{b}(X, Y). \quad \square \end{aligned}$$

Note that, for every  $g \in G$ ,  $\text{Ad}_g$  is an automorphism of  $\mathfrak{g}$ , so for  $X, Y, Z \in \mathfrak{g}$ ,  $\mathfrak{b}(\text{Ad}_{\exp(tX)}(Y), \text{Ad}_{\exp(tX)}(Z)) = \mathfrak{b}(Y, Z)$ . Differentiating this, we get another useful property of the Killing form.

**Corollary 1.3.** *For  $X, Y, Z \in \mathfrak{g}$ ,  $\mathfrak{b}(\text{ad}_X(Y), Z) + \mathfrak{b}(Y, \text{ad}_X(Z)) = 0$ .*

Even with just this information so far, the Killing form lets us prove very interesting things.

**Proposition 1.4.** *The Lie algebra  $\mathfrak{sl}_2\mathbb{R}$  is isomorphic to  $\mathfrak{o}(1, 2)$ .*

*Proof.* Consider the symmetric bilinear form  $-\mathfrak{h}$  on the 3-dimensional vector space  $\mathfrak{sl}_2\mathbb{R}$ . By Corollary 1.3, for  $X, Y, Z \in \mathfrak{sl}_2\mathbb{R}$ ,

$$-\mathfrak{h}(\text{ad}_X(Y), Z) - \mathfrak{h}(Y, \text{ad}_X(Z)) = 0,$$

so under the adjoint representation,  $\mathfrak{sl}_2\mathbb{R}$  maps into the Lie algebra  $\mathfrak{o}(-\mathfrak{h}) \approx \mathfrak{o}(1, 2)$ . Since the adjoint representation of  $\mathfrak{sl}_2\mathbb{R}$  is injective and  $\dim(\mathfrak{sl}_2\mathbb{R}) = 3 = \dim(\mathfrak{o}(1, 2))$ , this means that the adjoint representation gives an isomorphism of  $\mathfrak{sl}_2\mathbb{R}$  with  $\mathfrak{o}(1, 2)$ .  $\square$

An explicit realization of this isomorphism  $\rho : \mathfrak{sl}_2\mathbb{R} \rightarrow \mathfrak{o}(1, 2)$  is given by

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mapsto \begin{bmatrix} 0 & b+c & 2a \\ b+c & 0 & c-b \\ 2a & b-c & 0 \end{bmatrix},$$

with inverse  $\rho^{-1} : \mathfrak{o}(1, 2) \rightarrow \mathfrak{sl}_2\mathbb{R}$  given by

$$\begin{bmatrix} 0 & r & s \\ r & 0 & -t \\ s & t & 0 \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} s & r+t \\ r-t & -s \end{bmatrix}.$$

We'll use this to relate elements of  $\mathfrak{sl}_2\mathbb{R}$  to hyperbolic geometry.

## 2. CLASSIFICATION OF ELEMENTS OF $\mathfrak{sl}_2\mathbb{R}$

Elements of  $\mathfrak{sl}_2\mathbb{R}$  have a well-known classification using the terminology of conic sections: every  $X \in \mathfrak{sl}_2\mathbb{R}$  is either hyperbolic, parabolic, or elliptic.

**Definition 2.1.** Suppose  $X \in \mathfrak{sl}_2\mathbb{R}$ , viewed as a linear endomorphism of  $\mathbb{R}^2$ .

- If  $X$  is diagonalizable over  $\mathbb{R}$ , then we say that  $X$  is *hyperbolic*.
- If  $X$  is nilpotent, then we say that  $X$  is *parabolic*.
- If  $X$  has purely imaginary eigenvalues, then we say that  $X$  is *elliptic*.

It's not too difficult to see that every nonzero  $X \in \mathfrak{sl}_2\mathbb{R}$  falls into exactly one of these three categories: because  $X$  has trace 0 by definition, the complex eigenvalues of  $X$  must be  $\lambda$  and  $-\lambda$  for some  $\lambda \in \mathbb{C}$ . Since  $X$  is a real matrix, the eigenvalues must be complex conjugates of each other if they are not real, so if they are not real, then  $\lambda$  and  $-\lambda = \bar{\lambda}$  must be purely imaginary. If  $\lambda = 0 = -\lambda$  and  $X \neq 0$ , then  $X$  must be nilpotent. Finally, if  $\lambda$  and  $-\lambda$  are real and nonzero, then  $X$  is diagonalizable over  $\mathbb{R}$  by definition.

Of course, none of this explains why we're using this conic section terminology. Where does this terminology come from?

Recall that our model for the hyperbolic plane was  $(\mathrm{PO}(1, 2), \mathrm{O}(2))$ , so that elements of  $\mathfrak{o}(1, 2)$  determine one-parameter subgroups of hyperbolic isometries. We also had a projection map  $\mathrm{pr} : \mathbb{H}^2 \rightarrow \mathbb{R}^2$ , given by identifying  $\mathbb{H}^2 \cong \mathrm{PO}(1, 2)/\mathrm{O}(2)$  with the sheet through  $e_1$  of the hyperboloid  $Q^{-1}(1)$  for  $Q(ae_1 + be_2 + ce_3) = a^2 - b^2 - c^2$  and then projecting to the plane  $\langle e_2, e_3 \rangle$ , which allowed us to topologically identify  $\mathbb{H}^2$  with  $\mathbb{R}^2$ .

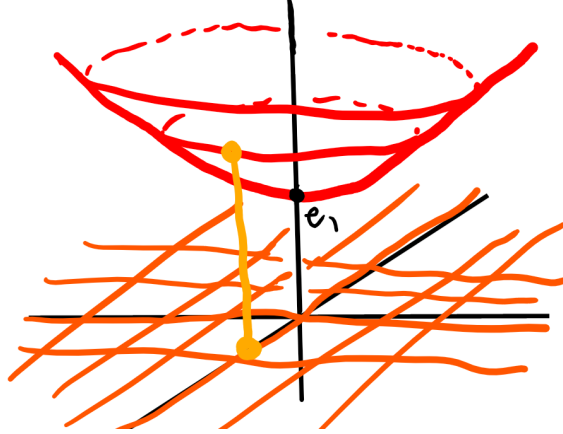


FIGURE 1. The map  $\mathrm{pr} : \mathbb{H}^2 \rightarrow \mathbb{R}^2$  identifies the point  $ae_1 + be_2 + ce_3 \in Q^{-1}(1)$  with the point  $\begin{bmatrix} b \\ c \end{bmatrix} \in \mathbb{R}^2$

Utilizing the isomorphism  $\rho : \mathfrak{sl}_2\mathbb{R} \rightarrow \mathfrak{o}(1, 2)$ , let us look at some one-parameter subgroups of hyperbolic isometries corresponding to elements of  $\mathfrak{sl}_2\mathbb{R}$ .

The parabolic element  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  of  $\mathfrak{sl}_2\mathbb{R}$  maps to

$$\rho\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

so

$$\exp(t\rho\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)) = \begin{pmatrix} 1 + \frac{t^2}{2} & t & -\frac{t^2}{2} \\ t & 1 & -t \\ \frac{t^2}{2} & t & 1 - \frac{t^2}{2} \end{pmatrix}.$$

Applying this to  $e_1$ , thought of as the identity coset of  $\mathrm{PO}(1, 2)/\mathrm{O}(2)$ , and looking at the image under the projection  $\mathrm{pr}$ , we get

$$\mathrm{pr}(\exp(t\rho\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)) \cdot e_1) = \begin{bmatrix} t \\ t^2/2 \end{bmatrix}.$$

In particular, the orbit of this one-parameter subgroup through  $e_1$  projects to a **parabola**! Indeed, all of its orbits on  $\mathbb{H}^2$  project to parabolas!

More generally, let's consider the element  $\frac{1}{2}\begin{bmatrix} 0 & r+1 \\ r-1 & 0 \end{bmatrix}$ . For  $r^2 > 1$ , this element has eigenvalues  $\lambda = \frac{1}{2}\sqrt{r^2 - 1}$  and  $-\lambda$ , so it is diagonalizable

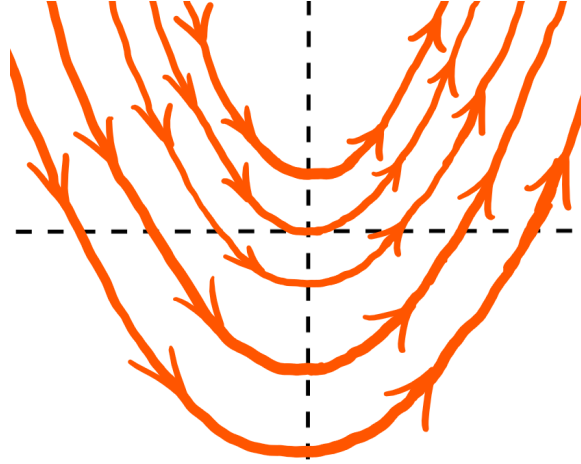


FIGURE 2. The orbits of the one-parameter subgroup  $\exp(t\rho(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}))$  all project to parabolas!

over  $\mathbb{R}$ , hence it is hyperbolic. In this case, we get corresponding one-parameter subgroups

$$\exp\left(\frac{t}{2}\rho\left(\begin{bmatrix} 0 & r+1 \\ r-1 & 0 \end{bmatrix}\right)\right) = \begin{pmatrix} \frac{r^2 \cosh(\sqrt{r^2-1}t)-1}{r^2-1} & r \frac{\sinh(\sqrt{r^2-1}t)}{\sqrt{r^2-1}} & -r \frac{\cosh(\sqrt{r^2-1}t)-1}{r^2-1} \\ r \frac{\sinh(\sqrt{r^2-1}t)}{\sqrt{r^2-1}} & \cosh(\sqrt{r^2-1}t) & -\frac{\sinh(\sqrt{r^2-1}t)}{\sqrt{r^2-1}} \\ r \frac{\cosh(\sqrt{r^2-1}t)-1}{r^2-1} & \frac{\sinh(\sqrt{r^2-1}t)}{\sqrt{r^2-1}} & \frac{r^2 - \cosh(\sqrt{r^2-1}t)}{r^2-1} \end{pmatrix},$$

and the orbit of this through  $e_1$  projects to (the connected component through 0 of) the **hyperbola** given by

$$\left(y + \frac{r}{r^2-1}\right)^2 - \frac{x^2}{r^2-1} = \left(\frac{r}{r^2-1}\right)^2.$$

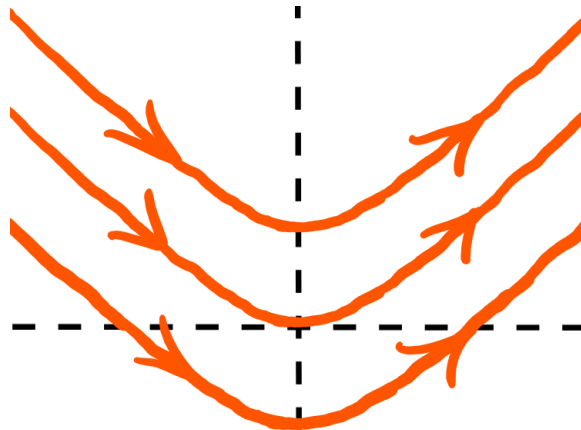


FIGURE 3. When  $r^2 > 1$ , the orbits of the one-parameter subgroup  $\exp(\frac{t}{2}\rho(\begin{bmatrix} 0 & r+1 \\ r-1 & 0 \end{bmatrix}))$  all project to hyperbolas with eccentricity  $|r|$

In fact, every orbit of this one-parameter subgroup projects to (a connected component of) a hyperbola with eccentricity<sup>1</sup>  $|r|$ .

Finally, as you may have guessed by now, the element  $\frac{1}{2} \begin{bmatrix} 0 & r+1 \\ r-1 & 0 \end{bmatrix}$  is elliptic for  $r^2 < 1$ . Such elements give us one-parameter subgroups

$$\exp\left(\frac{t}{2}\rho\left(\begin{bmatrix} 0 & r+1 \\ r-1 & 0 \end{bmatrix}\right)\right) = \begin{pmatrix} \frac{1-r^2 \cos(\sqrt{1-r^2}t)}{1-r^2} & r \frac{\sin(\sqrt{1-r^2}t)}{\sqrt{1-r^2}} & -r \frac{1-\cos(\sqrt{1-r^2}t)}{1-r^2} \\ r \frac{\sin(\sqrt{1-r^2}t)}{\sqrt{1-r^2}} & \cos(\sqrt{1-r^2}t) & -\frac{\sin(\sqrt{1-r^2}t)}{\sqrt{1-r^2}} \\ r \frac{1-\cos(\sqrt{1-r^2}t)}{1-r^2} & \frac{\sin(\sqrt{1-r^2}t)}{\sqrt{1-r^2}} & \frac{\cos(\sqrt{1-r^2}t)-r^2}{1-r^2} \end{pmatrix},$$

whose orbits project to **ellipses** of eccentricity  $|r|$  (except for the orbit that consists of the fixed point of the one-parameter subgroup). In particular, for  $r \neq 0$ , the orbit of  $\exp\left(\frac{t}{2}\rho\left(\begin{bmatrix} 0 & r+1 \\ r-1 & 0 \end{bmatrix}\right)\right)$  through  $e_1$  projects to the ellipse determined by the equation

$$\left(y - \frac{r}{1-r^2}\right)^2 + \frac{x^2}{1-r^2} = \left(\frac{r}{1-r^2}\right)^2.$$

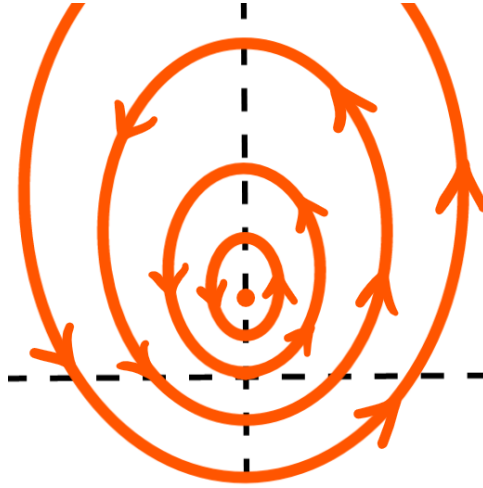


FIGURE 4. When  $r^2 < 1$ , the orbits of the one-parameter subgroup  $\exp\left(\frac{t}{2}\rho\left(\begin{bmatrix} 0 & r+1 \\ r-1 & 0 \end{bmatrix}\right)\right)$  all project to ellipses with eccentricity  $|r|$ , except for the one orbit corresponding to the fixed point

In general, elements of  $\mathfrak{sl}_2\mathbb{R}$  are elliptic, parabolic, or hyperbolic according to whether the orbits of their one-parameter subgroups project to conic sections of eccentricity in  $[0, 1)$ , equal to 1, or greater than 1, respectively.

<sup>1</sup>Recall that the *eccentricity* of a conic is the ratio of the distance of a given point from a point called the “focus” and the distance of that same point from a line called the “directrix”. The actual definition isn’t really that important though; the significance of the eccentricity is that it completely determines a conic section up to similarity transformations, with ellipses of eccentricity in  $[0, 1)$ , parabolas of eccentricity 1, and hyperbolas of (finite) eccentricity greater than 1.

It’s probably worth noting as well, as I remembered the night before the lecture, that elliptic and hyperbolic one-parameter subgroups for  $\mathrm{SL}_2\mathbb{R}$  also trace out ellipses and hyperbolas from the action on  $\mathbb{R}^2$ .

3. THE KILLING FORM ON  $\mathfrak{sl}_2\mathbb{R}$

Eccentricity gives us a parameter that uniquely determines a conic section up to similarity transformations. We would like something similar for elements of  $\mathfrak{sl}_2\mathbb{R}$ : a real number that completely characterizes an element of  $\mathfrak{sl}_2\mathbb{R}$  up to automorphism. As it turns out, the Killing form gives us such a parameter.

**Theorem 3.1.** *For nonzero  $X, Y \in \mathfrak{sl}_2\mathbb{R}$ , there is an automorphism  $\phi$  such that  $\phi(X) = Y$  if and only if  $\mathfrak{h}(X, X) = \mathfrak{h}(Y, Y)$ .*

*Proof.* This is actually much less daunting than it seems. To start, automorphisms of  $\mathfrak{sl}_2\mathbb{R}$  are exactly conjugations by elements of  $GL_2\mathbb{R}$ , so this is just a fancy way of saying that  $X$  and  $Y$  are conjugate over  $\mathbb{R}$  whenever  $\mathfrak{h}(X, X) = \mathfrak{h}(Y, Y)$ . To show this, we just find representatives of each conjugacy class and evaluate  $\mathfrak{h}$  on them; since  $\mathfrak{h}$  is invariant under automorphisms, the choice of representative does not matter.

The Jordan decomposition tells us that every nonzero  $X \in \mathfrak{sl}_2\mathbb{R}$  is conjugate over  $\mathbb{C}$  to precisely one matrix of the form  $\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , or  $\begin{bmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{bmatrix}$  for some  $\lambda > 0$ . Since real matrices that are conjugate over  $\mathbb{C}$  are conjugate over  $\mathbb{R}$  and  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  is conjugate to  $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$  over  $\mathbb{C}$ , this means that every nonzero  $X \in \mathfrak{sl}_2\mathbb{R}$  is conjugate over  $\mathbb{R}$  to exactly one element of the form  $\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , or  $\begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}$  for some  $\lambda > 0$ .

Thus, because we have  $\mathfrak{h}(\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}) = 8\lambda^2$ ,  $\mathfrak{h}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = 0$ , and  $\mathfrak{h}(\begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}) = -8\lambda^2$ , which never coincide for  $\lambda > 0$ , each conjugacy class is uniquely determined by the value of the Killing form.  $\square$

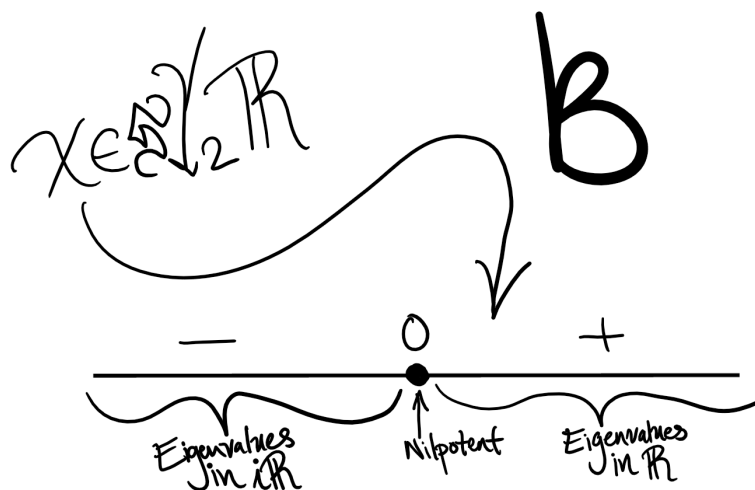


FIGURE 5. We can imagine  $\mathfrak{h}(X, X)$  for  $X \in \mathfrak{sl}_2\mathbb{R}$  to be a parameter describing  $X$  on a continuum where negative values correspond to diagonalizability over  $\mathbb{C}$  with imaginary eigenvalues, 0 corresponds to nilpotence, and positive values correspond to diagonalizability over  $\mathbb{R}$

From the above proof, we see that an element  $X \in \mathfrak{sl}_2\mathbb{R}$  is elliptic if and only if  $\mathfrak{h}(X, X) < 0$ , parabolic if and only if  $\mathfrak{h}(X, X) = 0$ , and hyperbolic if and only if  $\mathfrak{h}(X, X) > 0$ . In particular, we can imagine  $\mathfrak{h}(X, X)$  to be a parameter describing  $X$  on a continuum where negative values correspond to diagonalizability over  $\mathbb{C}$  with imaginary eigenvalues, 0 corresponds to nilpotence, and positive values correspond to diagonalizability over  $\mathbb{R}$ .

#### 4. WHAT DOES THE KILLING FORM TELL US?

For general Lie algebras  $\mathfrak{g}$ , the Killing form obviously isn't going to completely determine elements up to automorphism the way it does for  $\mathfrak{sl}_2\mathbb{R}$ . Nevertheless,  $\mathfrak{h}$  can still tell us a lot about how elements of  $\mathfrak{g}$  behave, if we look at it the right way.

To start, note that  $\mathfrak{h}(X, X) = \text{tr}(\text{ad}_X^2)$  is the sum of the squares of the eigenvalues of  $\text{ad}_X$ . This tells us, in particular, that if we want to understand  $\mathfrak{h}$ , then we need to look at elements, as well as notions like diagonalizability and nilpotence, from the perspective of the adjoint representation. Going back to the special case of  $\mathfrak{sl}_2\mathbb{R}$ , for example, we can reclassify elements in terms of the adjoint representation.

**Definition 4.1.** Suppose  $X \in \mathfrak{sl}_2\mathbb{R}$ .

- $X$  is *hyperbolic* if and only if  $\text{ad}_X$  is diagonalizable over  $\mathbb{R}$ .
- $X$  is *parabolic* if and only if it is ad-nilpotent.
- $X$  is *elliptic* if and only if  $\text{ad}_X$  is diagonalizable over  $\mathbb{C}$  with eigenvalues in  $i\mathbb{R}$ .

Even in a general real Lie algebra  $\mathfrak{g}$ , if  $\text{ad}_X$  is diagonalizable over  $\mathbb{R}$ , then we will have  $\mathfrak{h}(X, X) > 0$ . Of course, we won't necessarily get the converse as we do for  $\mathfrak{sl}_2\mathbb{R}$ , but if  $\mathfrak{h}(X, X) > 0$ , then we *can* say that the sum of the squares of the real parts of the eigenvalues of  $\text{ad}_X$  is bigger than the sum of the squares of the imaginary parts. In other words,  $\mathfrak{h}(X, X) > 0$  if and only if the real parts of the eigenvalues contribute the most to the behavior of  $\text{ad}_X$ , in which case we can think of it as “mostly scaling”.

Similarly, when  $\mathfrak{h}(X, X) < 0$ , the sum of the squares of the imaginary parts of the eigenvalues of  $\text{ad}_X$  is more than the sum of the squares of the real parts. In particular, if  $\text{ad}_X$  is diagonalizable over  $\mathbb{C}$  with all eigenvalues in  $i\mathbb{R}$ , then  $\mathfrak{h}(X, X) < 0$ , and while we can't get a true converse in general, we can think of  $\text{ad}_X$  in this case as being “mostly rotation”. Notably, we will have  $\mathfrak{h}(X, X) < 0$  whenever  $X$  comes from a subalgebra corresponding to a compact subgroup and  $X$  isn't central.

Finally, ad-nilpotent elements  $X$  will satisfy  $\mathfrak{h}(X, X) = 0$ . Again, unlike in the case of  $\mathfrak{sl}_2\mathbb{R}$ ,  $\mathfrak{h}(X, X) = 0$  doesn't necessarily guarantee that  $\text{ad}_X$  is nilpotent, but it does mean that the sum of the squares of the eigenvalues of  $\text{ad}_X$  is 0. We can kind of think of this as meaning that “the compact and scaling parts of  $\text{ad}_X$  cancel out”.



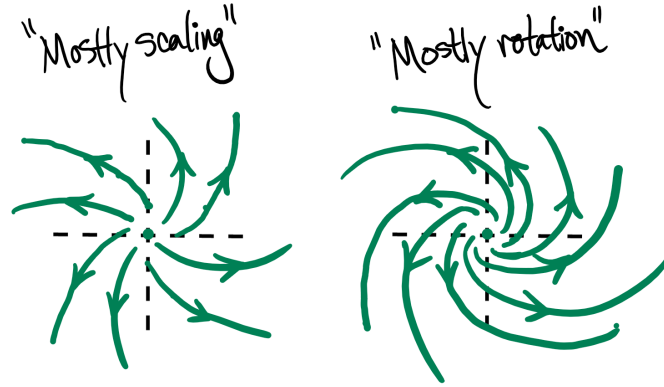


FIGURE 6. Attempted illustrations of the terms “mostly scaling” and “mostly rotation”

**Exercise.** Suppose  $K$  is a compact Lie group. What can we say about the Killing form on the Lie algebra  $\mathfrak{k}$ ? (Note that  $K$  could have non-trivial center.)

**Exercise.** Suppose  $N$  is a nilpotent Lie group. What can we say about the Killing form on the Lie algebra  $\mathfrak{n}$ ?

**Exercise.** Using that elements of the ideal of translations in  $\mathfrak{i}(2)$  are ad-nilpotent, describe the Killing form on  $\mathfrak{i}(2)$  without performing any computations.

Of course, we would also like to be able to say things about  $\mathfrak{h}(X, Y)$  for  $X \neq Y$ . Using polarization,

$$\mathfrak{h}(X, Y) = \frac{1}{2}(\mathfrak{h}(X + Y, X + Y) - \mathfrak{h}(X, X) - \mathfrak{h}(Y, Y)).$$

This is particularly useful when  $X$  and  $Y$  are ad-nilpotent, in which case  $\mathfrak{h}(X, Y) = \frac{1}{2}\mathfrak{h}(X + Y, X + Y)$ . Thus, for ad-nilpotent  $X$  and  $Y$ ,  $\mathfrak{h}(X, Y) > 0$  when  $\text{ad}_{X+Y}$  is “mostly scaling” and  $\mathfrak{h}(X, Y) < 0$  when  $\text{ad}_{X+Y}$  is “mostly rotation”. For example,  $\mathfrak{h}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) > 0$  because  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  is “mostly scaling”, and  $\mathfrak{h}(\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) < 0$  because  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  is “mostly rotation”.

Next time, we will focus on  $\mathfrak{g}$  where  $\mathfrak{h}$  is nondegenerate, in which case we say that  $\mathfrak{g}$  is *semisimple*. For semisimple Lie algebras, the behavior of  $\mathfrak{h}$  described above suggests a particular form for  $\mathfrak{g}$ : there should be an ad-diagonalizable part together with ad-nilpotent elements occurring in pairs on which the Killing form is nonzero.